

# Noncommutative localization in topology

Andrew Ranicki

## Introduction

The topological applications of the Cohn noncommutative localization considered in this paper deal with spaces (especially manifolds) with infinite fundamental group, and involve localizations of infinite group rings and related triangular matrix rings. Algebraists have usually considered noncommutative localization of rather better behaved rings, so the topological applications require new algebraic techniques.

Part 1 is a brief survey of the applications of noncommutative localization to topology: finitely dominated spaces, codimension 1 and 2 embeddings (knots and links), homology surgery theory, open book decompositions and circle-valued Morse theory. These applications involve chain complexes and the algebraic  $K$ - and  $L$ -theory of the noncommutative localization of group rings.

Part 2 is a report on work on chain complexes over generalized free products and the related algebraic  $K$ - and  $L$ -theory, from the point of view of noncommutative localization of triangular matrix rings. Following Bergman and Schofield, a generalized free product of rings can be constructed as a noncommutative localization of a triangular matrix ring. The novelty here is the explicit connection to the algebraic topology of manifolds with a generalized free product structure realized by a codimension 1 submanifold, leading to noncommutative localization proofs of the results of Waldhausen and Cappell on the algebraic  $K$ - and  $L$ -theory of generalized free products. In a sense, this is more in the nature of an application of topology to noncommutative localization! But this algebra has in turn topological applications, since in dimensions  $\geq 5$  the surgery classification of manifolds within a homotopy type reduces to algebra.

## Part 1. A survey of applications

We start by recalling the universal noncommutative localization of P.M.Cohn [5]. Let  $A$  be a ring, and let  $\Sigma = \{s : P \rightarrow Q\}$  be a set of morphism of f.g. projective  $A$ -modules. A ring morphism  $A \rightarrow R$  is  $\Sigma$ -*inverting* if for every  $s \in \Sigma$  the induced morphism of f.g. projective  $R$ -modules  $1 \otimes s : R \otimes_A P \rightarrow R \otimes_A Q$  is an isomorphism. The noncommutative localization  $A \rightarrow \Sigma^{-1}A$  is  $\Sigma$ -inverting, and has the universal property that any  $\Sigma$ -inverting ring morphism  $A \rightarrow R$  has a unique factorization  $A \rightarrow \Sigma^{-1}A \rightarrow R$ . The applications to topology involve homology with coefficients in a noncommutative localization  $\Sigma^{-1}A$ .

Homology with coefficients is defined as follows. Let  $X$  be a connected topological space with universal cover  $\tilde{X}$ , and let the fundamental group  $\pi_1(X)$  act on the left of  $\tilde{X}$ , so that the (singular) chain complex  $S(\tilde{X})$  is a free left  $\mathbb{Z}[\pi_1(X)]$ -module complex. Given a morphism of rings  $\mathcal{F} : \mathbb{Z}[\pi_1(X)] \rightarrow \Lambda$  define the  $\Lambda$ -*coefficient homology of  $X$*  to be

$$H_*(X; \Lambda) = H_*(\Lambda \otimes_{\mathbb{Z}[\pi_1(X)]} S(\tilde{X})) .$$

If  $X$  is a  $CW$  complex then  $S(\tilde{X})$  is chain equivalent to the cellular free  $\mathbb{Z}[\pi_1(X)]$ -module chain complex  $C(\tilde{X})$  with one generator in degree  $r$  for each  $r$ -cell of  $X$ , and

$$H_*(X; \Lambda) = H_*(\Lambda \otimes_{\mathbb{Z}[\pi_1(X)]} C(\tilde{X})) .$$

### 1.1 Finite domination

A topological space  $X$  is *finitely dominated* if there exist a finite  $CW$  complex  $K$ , maps  $f : X \rightarrow K$ ,  $g : K \rightarrow X$  and a homotopy  $gf \simeq 1 : X \rightarrow X$ . The finiteness obstruction of Wall [31] is a reduced projective class  $[X] \in \tilde{K}_0(\mathbb{Z}[\pi_1(X)])$  such that  $[X] = 0$  if and only if  $X$  is homotopy equivalent to a finite  $CW$  complex.

In the applications of the finiteness obstruction to manifold topology  $X = \overline{M}$  is an infinite cyclic cover of a compact manifold  $M$  – see Chapter 17 of Hughes and Ranicki [13] for the geometric wrapping up procedure which shows that in dimension  $\geq 5$  every tame manifold end has a neighbourhood which is a finitely dominated infinite cyclic cover  $\overline{M}$  of a compact manifold  $M$ . Let  $f : M \rightarrow S^1$  be a classifying map, so that  $\overline{M} = f^*\mathbb{R}$ , and let  $\overline{M}^+ = f^*\mathbb{R}^+$ . The finiteness obstruction  $[\overline{M}^+] \in \tilde{K}_0(\mathbb{Z}[\pi_1(\overline{M})])$  is the end

obstruction of Siebenmann [27], such that  $[\overline{M}^+] = 0$  if and only if the tame end can be closed, i.e. compactified by a manifold with boundary.

Given a ring  $A$  let  $\Omega$  be the set of square matrices  $\omega \in M_r(A[z, z^{-1}])$  over the Laurent polynomial extension  $A[z, z^{-1}]$  such that the  $A$ -module

$$P = \text{coker}(\omega : A[z, z^{-1}]^r \rightarrow A[z, z^{-1}]^r)$$

is f.g. projective. The noncommutative Fredholm localization  $\Omega^{-1}A[z, z^{-1}]$  has the universal property that a finite f.g. free  $A[z, z^{-1}]$ -module chain complex  $C$  is  $A$ -module chain equivalent to a finite f.g. projective  $A$ -module chain complex if and only if  $H_*(\Omega^{-1}C) = 0$  (Ranicki [21, Proposition 13.9]), with  $\Omega^{-1}C = \Omega^{-1}A[z, z^{-1}] \otimes_{A[z, z^{-1}]} C$ .

Let  $M$  be a connected finite  $CW$  complex with a connected infinite cyclic cover  $\overline{M}$ . The fundamental group  $\pi_1(M)$  fits into an extension

$$\{1\} \rightarrow \pi_1(\overline{M}) \rightarrow \pi_1(M) \rightarrow \mathbb{Z} \rightarrow \{1\}$$

and  $\mathbb{Z}[\pi_1(M)]$  is a twisted Laurent polynomial extension

$$\mathbb{Z}[\pi_1(M)] = \mathbb{Z}[\pi_1(\overline{M})]_\alpha[z, z^{-1}]$$

with

$$\alpha : \pi_1(\overline{M}) \rightarrow \pi_1(\overline{M}) ; g \mapsto z^{-1}gz$$

the monodromy automorphism. For the sake of simplicity only the untwisted case  $\alpha = 1$  will be considered here, so that  $\pi_1(M) = \pi_1(\overline{M}) \times \mathbb{Z}$ . The infinite cyclic cover  $\overline{M}$  is finitely dominated if and only if  $H_*(M; \Omega^{-1}\mathbb{Z}[\pi_1(M)]) = 0$ , with  $A = \mathbb{Z}[\pi_1(\overline{M})]$  and  $\mathbb{Z}[\pi_1(M)] = A[z, z^{-1}]$ . The Farrell-Siebenmann obstruction  $\Phi(M) \in \text{Wh}(\pi_1(M))$  of an  $n$ -dimensional manifold  $M$  with finitely dominated infinite cyclic cover  $\overline{M}$  is such that  $\Phi(M) = 0$  if (and for  $n \geq 6$  only if)  $M$  is a fibre bundle over  $S^1$  – see [21, Proposition 15.16] for the expression of  $\Phi(M)$  in terms of the  $\Omega^{-1}\mathbb{Z}[\pi_1(M)]$ -coefficient Reidemeister-Whitehead torsion

$$\tau(M; \Omega^{-1}\mathbb{Z}[\pi_1(M)]) = \tau(\Omega^{-1}C(\widetilde{M})) \in K_1(\Omega^{-1}\mathbb{Z}[\pi_1(M)]) .$$

## 1.2 Codimension 1 splitting

Surgery theory asks whether a homotopy equivalence of manifolds is homotopic (or  $h$ -cobordant) to a homeomorphism – in general, the answer is no. There are obstructions in the topological  $K$ -theory of vector bundles, in the

algebraic  $K$ -theory of modules and in the algebraic  $L$ -theory of quadratic forms. The algebraic  $K$ -theory obstruction lives in the Whitehead group  $Wh(\pi)$  of the fundamental group  $\pi$ . The  $L$ -theory obstruction lives in one of the surgery groups  $L_*(\mathbb{Z}[\pi])$  of Wall [32], and is defined when the topological and algebraic  $K$ -theory obstructions vanish. The groups  $L_*(\Lambda)$  are defined for any ring with involution  $\Lambda$  to be the generalized Witt groups of nonsingular quadratic forms over  $\Lambda$ . For manifolds of dimension  $\geq 5$  the vanishing of the algebraic obstructions is both a necessary and sufficient condition for deforming a homotopy equivalence to a homeomorphism. See Ranicki [20] for the reduction of the Browder-Novikov-Sullivan-Wall surgery theory to algebra.

A homotopy equivalence of  $m$ -dimensional manifolds  $f : M' \rightarrow M$  splits along a submanifold  $N^n \subset M^m$  if  $f$  is homotopic to a map (also denoted by  $f$ ) such that  $N' = f^{-1}(N) \subset M'$  is also a submanifold, and the restriction  $f| : N' \rightarrow N$  is also a homotopy equivalence. For codimension  $m - n \geq 3$  the splitting obstruction is just the ordinary surgery obstruction  $\sigma_*(f) \in L_m(\mathbb{Z}[\pi_1(N)])$ . For codimension  $m - n = 1, 2$  the splitting obstructions involve the interplay of the knotting properties of codimension  $(m - n)$  submanifolds and Mayer-Vietoris-type decompositions of the algebraic  $K$ - and  $L$ -groups of  $\mathbb{Z}[\pi_1(M)]$  in terms of the groups of  $\mathbb{Z}[\pi_1(N)]$ ,  $\mathbb{Z}[\pi_1(M \setminus N)]$ .

In the case  $m - n = 1$   $\pi_1(M)$  is a generalized free product, i.e. either an amalgamated free product or an  $HNN$  extension, by the Seifert-van Kampen theorem. Codimension 1 splitting theorems and the algebraic  $K$ - and  $L$ -theory of generalized free products are a major ingredient of high-dimensional manifold topology, featuring in the work of Stallings, Browder, Novikov, Wall, Siebenmann, Farrell, Hsiang, Shaneson, Casson, Waldhausen, Cappell, . . . , and the author. Noncommutative localization provides a systematic development of this algebra, using the intuition afforded by the topological applications – see Part 2 below for a more detailed discussion.

### 1.3 Homology surgery theory

For a morphism of rings with involution  $\mathcal{F} : \mathbb{Z}[\pi] \rightarrow \Lambda$  Cappell and Shaneson [3] considered the problem of whether a  $\Lambda$ -coefficient homology equivalence of manifolds with fundamental group  $\pi$  is  $H$ -cobordant to a homeomorphism. Again, the answer is no in general, with obstructions in the topological  $K$ -theory of vector bundles and in the homology surgery groups  $\Gamma_*(\mathcal{F})$ , which are generalized Witt groups of  $\Lambda$ -nonsingular quadratic forms over

$\mathbb{Z}[\pi]$ . Vogel [28], [29] identified the  $\Lambda$ -coefficient homology surgery groups with the ordinary  $L$ -groups of the localization  $\Sigma^{-1}\mathbb{Z}[\pi]$  of  $\mathbb{Z}[\pi]$  inverting the set  $\Sigma$  of  $\Lambda$ -invertible square matrices over  $\mathbb{Z}[\pi]$

$$\Gamma_*(\mathcal{F}) = L_*(\Sigma^{-1}\mathbb{Z}[\pi]) ,$$

and identified the relative  $L$ -groups  $L_*(\mathbb{Z}[\pi] \rightarrow \Sigma^{-1}\mathbb{Z}[\pi])$  in the localization exact sequence

$$\cdots \rightarrow L_n(\mathbb{Z}[\pi]) \rightarrow L_n(\Sigma^{-1}\mathbb{Z}[\pi]) \rightarrow L_n(\mathbb{Z}[\pi] \rightarrow \Sigma^{-1}\mathbb{Z}[\pi]) \rightarrow L_{n-1}(\mathbb{Z}[\pi]) \rightarrow \cdots$$

with generalized Witt groups  $L_*(\mathbb{Z}[\pi], \Sigma)$  of nonsingular  $\Sigma^{-1}\mathbb{Z}[\pi]/\mathbb{Z}[\pi]$ -valued quadratic linking forms on  $\Sigma$ -torsion  $\mathbb{Z}[\pi]$ -modules of homological dimension 1.

## 1.4 Codimension 2 embeddings

Suppose given a codimension 2 embedding  $N^n \subset M^{n+2}$  such as a knot or link. Let  $\Sigma^{-1}A$  be the localization of  $A = \mathbb{Z}[\pi_1(M \setminus N)]$  inverting the set  $\Sigma$  of matrices over  $A$  which become invertible over  $\mathbb{Z}[\pi_1(M)]$ . By Alexander duality the  $\Sigma^{-1}A$ -coefficient homology modules

$$H_*(M \setminus N; \Sigma^{-1}A) \cong H^{n+2-*}(M, N; \Sigma^{-1}A) \quad (* \neq 0, n+2)$$

are determined by the homotopy class of the inclusion  $N \subset M$ . The  $A$ -coefficient homology groups  $H_*(M \setminus N; A)$  and their Poincaré duality properties reflect more subtle invariants of  $N \subset M$  such as knotting. See Ranicki [21] for a general account of high-dimensional codimension 2 embedding theory, including some of the applications of noncommutative localization.

## 1.5 Open books

An  $(n+2)$ -dimensional manifold  $M^{n+2}$  is an *open book* if there exists a codimension 2 submanifold  $N^n \subset M^{n+2}$  such that the complement  $M \setminus N$  is a fibre bundle over  $S^1$ . Every odd-dimensional manifold is an open book. Quinn [17] showed that for  $k \geq 2$  a  $(2k+2)$ -dimensional manifold  $M$  is an open book if and only if an asymmetric form over  $\mathbb{Z}[\pi_1(M)]$  associated to  $M$  represents 0 in the Witt group. This obstruction was identified in Ranicki [21] with an element in the  $L$ -group  $L_{2k+2}(\Omega^{-1}\mathbb{Z}[\pi_1(M)][z, z^{-1}])$  of the Fredholm localization of  $\mathbb{Z}[\pi_1(M)][z, z^{-1}]$  (cf. section 1.1 above).

## 1.6 Boundary link cobordism

An  $n$ -dimensional  $\mu$ -component boundary link is a codimension 2 embedding

$$N^n = \bigcup_{\mu} S^n \subset M^{n+2} = S^{n+2}$$

with a  $\mu$ -component Seifert surface, in which case the fundamental group of the complement  $X = M \setminus N$  has a compatible surjection  $\pi_1(X) \rightarrow F_{\mu}$  onto the free group on  $\mu$  generators. Duval [8] used the work of Cappell and Shaneson [4] and Vogel [29] to identify the cobordism group of  $n$ -dimensional  $\mu$ -component boundary links for  $n \geq 2$  with the relative  $L$ -group  $L_{n+3}(\mathbb{Z}[F_{\mu}], \Sigma)$  in the localization exact sequence

$$\cdots \rightarrow L_{n+3}(\mathbb{Z}[F_{\mu}]) \rightarrow L_{n+3}(\Sigma^{-1}\mathbb{Z}[F_{\mu}]) \rightarrow L_{n+3}(\mathbb{Z}[F_{\mu}], \Sigma) \rightarrow L_{n+2}(\mathbb{Z}[F_{\mu}]) \rightarrow \cdots$$

with  $\Sigma$  the set of  $\mathbb{Z}$ -invertible square matrices over  $\mathbb{Z}[F_{\mu}]$ . The even-dimensional boundary link cobordism groups are  $L_{2k+1}(\mathbb{Z}[F_{\mu}], \Sigma) = 0$ . The cobordism class in  $L_{2k+2}(\mathbb{Z}[F_{\mu}], \Sigma)$  of a  $(2k-1)$ -dimensional  $\mu$ -component boundary link  $\cup_{\mu} S^{2k-1} \subset S^{2k+1}$  was identified with the Witt class of a  $\Sigma^{-1}\mathbb{Z}[F_{\mu}]/\mathbb{Z}[F_{\mu}]$ -valued nonsingular  $(-1)^{k+1}$ -quadratic linking form on  $H_k(X; \mathbb{Z}[F_{\mu}])$ , generalizing the Blanchfield pairing on the homology of the infinite cyclic cover of a knot. The localization  $\Sigma^{-1}\mathbb{Z}[F_{\mu}]$  was identified by Dicks and Sontag [7] and Farber and Vogel [11] with a ring of rational functions in  $\mu$  noncommuting variables. The high odd-dimensional boundary link cobordism groups  $L_{2k+2}(\mathbb{Z}[F_{\mu}], \Sigma)$  have been computed by Sheiham [26].

## 1.7 Circle-valued Morse theory

Novikov [15] proposed the study of the critical points of Morse functions  $f : M \rightarrow S^1$  on compact manifolds  $M$ . The ‘Novikov complex’  $C(M, f)$  over  $\mathbb{Z}((z)) = \mathbb{Z}[[z]][z^{-1}]$  has one generator for each critical point of  $f$ , and the ‘Novikov homology’

$$H_*(C(M, f)) = H_*(M; \mathbb{Z}((z)))$$

provides lower bounds on the number of critical points of Morse functions in the homotopy class of  $f$ , generalizing the inequalities of the classical Morse theory of real-valued functions  $M \rightarrow \mathbb{R}$ . Suppose given a Morse function  $f : M \rightarrow S^1$  with  $\overline{M} = f^*\mathbb{R}$  such that  $\pi_1(M) = \pi_1(\overline{M}) \times \mathbb{Z}$  (for the sake of simplicity). Let  $\Sigma$  be the set of square matrices over  $\mathbb{Z}[\pi_1(\overline{M})][z]$  which

become invertible over  $\mathbb{Z}[\pi_1(\overline{M})]$  under the augmentation  $z \mapsto 0$ . There is a natural morphism from the localization to the completion

$$\Sigma^{-1}\mathbb{Z}[\pi_1(M)] \rightarrow \widehat{\mathbb{Z}[\pi_1(M)]} = \mathbb{Z}[\pi_1(\overline{M})][[z]][z^{-1}]$$

which is an injection if  $\pi_1(M)$  is abelian or  $F_\mu$  (Dicks and Sontag [7], Farber and Vogel [11]), but may not be an injection in general (Sheiham [25]). See Pajitnov [16], Farber and Ranicki [10], Ranicki [22], and Cornea and Ranicki [6] for the construction and properties of Novikov complexes of  $f$  over  $\widehat{\mathbb{Z}[\pi_1(M)]}$  and  $\Sigma^{-1}\mathbb{Z}[\pi_1(M)]$ . Naturally, noncommutative localization also features in the more general Morse theory of closed 1-forms – see Novikov [15] and Farber [9].

### 1.8 3- and 4-dimensional manifolds

See Garoufalidis and Kriker [12], Quinn [18] for applications of noncommutative localization in the topology of 3- and 4-dimensional manifolds.

## Part 2. The algebraic K- and L-theory of generalized free products via noncommutative localization

A generalized free product of groups (or rings) is either an amalgamated free product or an  $HNN$  extension. The expressions of Schofield [24] of generalized free products as noncommutative localizations of triangular matrix rings combine with the localization exact sequences of Neeman and Ranicki [14] to provide more systematic proofs of the Mayer-Vietoris decompositions of Waldhausen [30] and Cappell [2] of the algebraic  $K$ - and  $L$ -theory of generalized free products. The topological motivation for these proofs comes from a noncommutative localization interpretation of the Seifert-van Kampen and Mayer-Vietoris theorems. If  $(M, N \subseteq M)$  is a two-sided pair of connected  $CW$  complexes the fundamental group  $\pi_1(M)$  is a generalized free product: an amalgamated free product if  $N$  separates  $M$ , and an  $HNN$  extension otherwise. The morphisms  $\pi_1(N) \rightarrow \pi_1(M \setminus N)$  determine a triangular  $k \times k$  matrix ring  $A$  with universal localization the full  $k \times k$  matrix ring  $\Sigma^{-1}A = M_k(\mathbb{Z}[\pi_1(M)])$  ( $k = 3$  in the separating case,  $k = 2$  in the non-separating case), such that the corresponding presentations of the  $\mathbb{Z}[\pi_1(M)]$ -module chain complex  $C(\widetilde{M})$  of the universal cover  $\widetilde{M}$  is the

assembly of an  $A$ -module chain complex constructed from the chain complexes  $C(\widetilde{N})$ ,  $C(\widetilde{M \setminus N})$  of the universal covers  $\widetilde{N}$ ,  $\widetilde{M \setminus N}$  of  $N$ ,  $M \setminus N$ . The two cases will be considered separately, in sections 2.3, 2.4.

## 2.1 The algebraic $K$ -theory of a noncommutative localization

Given an injective noncommutative localization  $A \rightarrow \Sigma^{-1}A$  let  $H(A, \Sigma)$  be the exact category of homological dimension 1  $A$ -modules  $T$  which admit a f.g. projective  $A$ -module resolution

$$0 \longrightarrow P \xrightarrow{s} Q \longrightarrow T \longrightarrow 0$$

such that  $1 \otimes s : \Sigma^{-1}P \rightarrow \Sigma^{-1}Q$  is an  $\Sigma^{-1}A$ -module isomorphism. The algebraic  $K$ -theory localization exact sequence of Schofield [24, Theorem 4.12]

$$K_1(A) \rightarrow K_1(\Sigma^{-1}A) \rightarrow K_1(A, \Sigma) \rightarrow K_0(A) \rightarrow K_0(\Sigma^{-1}A)$$

was obtained for any injective noncommutative localization  $A \rightarrow \Sigma^{-1}A$ , with  $K_1(A, \Sigma) = K_0(H(A, \Sigma))$ . Neeman and Ranicki [14] proved that if  $A \rightarrow \Sigma^{-1}A$  is injective and ‘stably flat’

$$\mathrm{Tor}_i^A(\Sigma^{-1}A, \Sigma^{-1}A) = 0 \quad (i \geq 1)$$

then

- (i)  $\Sigma^{-1}A$  has the chain complex lifting property : every finite f.g. free  $\Sigma^{-1}A$ -module chain complex  $C$  is chain equivalent to  $\Sigma^{-1}B$  for a finite f.g. projective  $A$ -module chain complex  $B$ ,
- (ii) the localization exact sequence extends to the higher  $K$ -groups

$$\cdots \rightarrow K_n(A) \rightarrow K_n(\Sigma^{-1}A) \rightarrow K_n(A, \Sigma) \rightarrow K_{n-1}(A) \rightarrow \cdots \rightarrow K_0(\Sigma^{-1}A)$$

with  $K_n(A, \Sigma) = K_{n-1}(H(A, \Sigma))$ .



## 2.2 Matrix rings

The amalgamated free product of rings and the  $HNN$  construction are special cases of the following type of noncommutative localization of triangular matrix rings.

Given rings  $A_1, A_2$  and an  $(A_1, A_2)$ -bimodule  $B$  define the triangular  $2 \times 2$  matrix ring

$$A = \begin{pmatrix} A_1 & B \\ 0 & A_2 \end{pmatrix} .$$

An  $A$ -module can be written as

$$M = \begin{pmatrix} M_1 \\ M_2 \end{pmatrix}$$

with  $M_1$  an  $A_1$ -module,  $M_2$  an  $A_2$ -module, together with an  $A_1$ -module morphism  $B \otimes_{A_2} M_2 \rightarrow M_1$ . The injection

$$A_1 \times A_2 \rightarrow A ; (a_1, a_2) \mapsto \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}$$

induces isomorphisms of algebraic  $K$ -groups

$$K_*(A_1) \oplus K_*(A_2) \cong K_*(A) .$$

The columns of  $A$  are f.g. projective  $A$ -modules

$$P_1 = \begin{pmatrix} A_1 \\ 0 \end{pmatrix} , P_2 = \begin{pmatrix} B \\ A_2 \end{pmatrix}$$

such that

$$\begin{aligned} P_1 \oplus P_2 &= A , \operatorname{Hom}_A(P_i, P_i) = A_i \ (i = 1, 2) , \\ \operatorname{Hom}_A(P_1, P_2) &= B , \operatorname{Hom}_A(P_2, P_1) = 0 . \end{aligned}$$

The noncommutative localization of  $A$  inverting a non-empty subset  $\Sigma \subseteq \operatorname{Hom}_A(P_1, P_2) = B$  is the  $2 \times 2$  matrix ring

$$\Sigma^{-1}A = M_2(C) = \begin{pmatrix} C & C \\ C & C \end{pmatrix}$$

with  $C$  the endomorphism ring of the induced f.g. projective  $\Sigma^{-1}A$ -module  $\Sigma^{-1}P_1 \cong \Sigma^{-1}P_2$ . The Morita equivalence

$$\{\Sigma^{-1}A\text{-modules}\} \rightarrow \{C\text{-modules}\} ; L \mapsto (C \ C) \otimes_{\Sigma^{-1}A} L$$

induces isomorphisms in algebraic  $K$ -theory

$$K_*(M_2(C)) \cong K_*(C) .$$

The composite of the functor

$$\{A\text{-modules}\} \rightarrow \{\Sigma^{-1}A\text{-modules}\} ; M \mapsto \Sigma^{-1}M = \Sigma^{-1}A \otimes_A M$$

and the Morita equivalence is the *assembly* functor

$$\begin{aligned} \{A\text{-modules}\} &\rightarrow \{C\text{-modules}\} ; \\ M = \begin{pmatrix} M_1 \\ M_2 \end{pmatrix} &\mapsto (C \ C) \otimes_A M \\ &= \text{coker}(C \otimes_{A_1} B \otimes_{A_2} M_2 \rightarrow C \otimes_{A_1} M_1 \oplus C \otimes_{A_2} M_2) \end{aligned}$$

inducing the morphisms

$$K_*(A) = K_*(A_1) \oplus K_*(A_2) \rightarrow K_*(\Sigma^{-1}A) = K_*(C)$$

in the algebraic  $K$ -theory localization exact sequence.

There are evident generalizations to  $k \times k$  matrix rings for any  $k \geq 2$ .

### 2.3 $HNN$ extensions

The  $HNN$  extension  $R *_{\alpha, \beta} \{z\}$  is defined for any ring morphisms  $\alpha, \beta : S \rightarrow R$ , with

$$\alpha(s)z = z\beta(s) \in R *_{\alpha, \beta} \{z\} \quad (s \in S) .$$

Define the triangular  $2 \times 2$  matrix ring

$$A = \begin{pmatrix} R & R_\alpha \oplus R_\beta \\ 0 & S \end{pmatrix}$$

with  $R_\alpha$  the  $(R, S)$ -bimodule  $R$  with  $S$  acting on  $R$  via  $\alpha$ , and similarly for  $R_\beta$ . Let  $\Sigma = \{\sigma_1, \sigma_2\} \subset \text{Hom}_A(P_1, P_2)$ , with

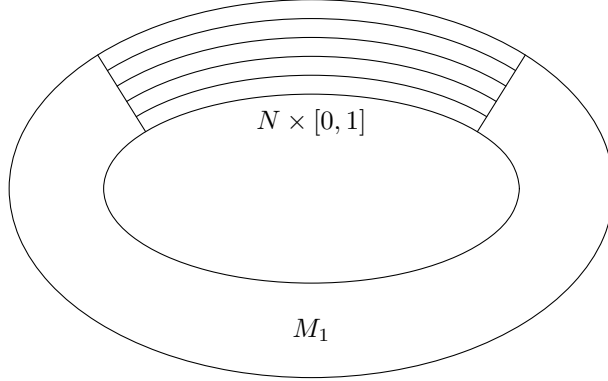
$$\sigma_1 = \begin{pmatrix} (1, 0) \\ 0 \end{pmatrix} , \sigma_2 = \begin{pmatrix} (0, 1) \\ 0 \end{pmatrix} : P_1 = \begin{pmatrix} R \\ 0 \end{pmatrix} \rightarrow P_2 = \begin{pmatrix} R_\alpha \oplus R_\beta \\ S \end{pmatrix} .$$

The  $A$ -modules  $P_1, P_2$  are f.g. projective since  $P_1 \oplus P_2 = A$ . Theorem 13.1 of [24] identifies

$$\Sigma^{-1}A = M_2(R *_{\alpha, \beta} \{z\}) .$$

*Example* Let  $(M, N \subseteq M)$  be a non-separating pair of connected  $CW$  complexes such that  $N$  is two-sided in  $M$  (i.e. has a neighbourhood  $N \times [0, 1] \subseteq M$ ) with  $M \setminus N = M_1$  connected

$$M = M_1 \cup_{N \times \{0,1\}} N \times [0, 1]$$



By the Seifert-van Kampen theorem, the fundamental group  $\pi_1(M)$  is the  $HNN$  extension determined by the morphisms  $\alpha, \beta : \pi_1(N) \rightarrow \pi_1(M_1)$  induced by the inclusions  $N \times \{0\} \rightarrow M_1, N \times \{1\} \rightarrow M_1$

$$\pi_1(M) = \pi_1(M_1) *_{\alpha, \beta} \{z\} ,$$

so that

$$\mathbb{Z}[\pi_1(M)] = \mathbb{Z}[\pi_1(M_1)] *_{\alpha, \beta} \{z\} .$$

As above, define a triangular  $2 \times 2$  matrix ring

$$A = \begin{pmatrix} \mathbb{Z}[\pi_1(N)] & \mathbb{Z}[\pi_1(M_1)]_\alpha \oplus \mathbb{Z}[\pi_1(M_1)]_\beta \\ 0 & \mathbb{Z}[\pi_1(M)] \end{pmatrix}$$

with noncommutative localization

$$\Sigma^{-1}A = M_2(\mathbb{Z}[\pi_1(M_1)] *_{\alpha, \beta} \{z\}) = M_2(\mathbb{Z}[\pi_1(M)]) .$$

Assume that  $\pi_1(N) \rightarrow \pi_1(M)$  is injective, so that the morphisms  $\alpha, \beta$  are injective, and the universal cover  $\widetilde{M}$  is a union

$$\widetilde{M} = \bigcup_{g \in [\pi_1(M) : \pi_1(M_1)]} g\widetilde{M}_1$$

of translates of the universal cover  $\widetilde{M}_1$  of  $M_1$ , and

$$g_1 \widetilde{M}_1 \cap g_2 \widetilde{M}_1 = \begin{cases} h\widetilde{N} & \text{if } g_1 \cap g_2 z = h \in [\pi_1(M) : \pi_1(N)] \\ g_1 \widetilde{M}_1 & \text{if } g_1 = g_2 \\ \emptyset & \text{if } g_1 \neq g_2 \text{ and } g_1 \cap g_2 z = \emptyset \end{cases}$$

with  $h\widetilde{N}$  the translates of the universal cover  $\widetilde{N}$  of  $N$ . In the diagram it is assumed that  $\alpha, \beta$  are isomorphisms

$\widetilde{M}$	$z^{-2}\widetilde{M}_1$	$z^{-1}\widetilde{M}_1$	$\widetilde{M}_1$	$z\widetilde{M}_1$	$z^2\widetilde{M}_1$
	$z^{-1}\widetilde{N}$	$\widetilde{N}$	$z\widetilde{N}$	$z^2\widetilde{N}$	

The cellular f.g. free chain complexes  $C(\widetilde{M}_1), C(\widetilde{N})$  are related by  $\mathbb{Z}[\pi_1(M_1)]$ -module chain maps

$$\begin{aligned} i_\alpha &: \mathbb{Z}[\pi_1(M_1)]_\alpha \otimes_{\mathbb{Z}[\pi_1(N)]} C(\widetilde{N}) \rightarrow C(\widetilde{M}_1) , \\ i_\beta &: \mathbb{Z}[\pi_1(M_1)]_\beta \otimes_{\mathbb{Z}[\pi_1(N)]} C(\widetilde{N}) \rightarrow C(\widetilde{M}_1) \end{aligned}$$

defining a f.g. projective  $A$ -module chain complex  $\begin{pmatrix} C(\widetilde{M}_1) \\ C(\widetilde{N}) \end{pmatrix}$  with assembly the cellular f.g. free  $\mathbb{Z}[\pi_1(M)]$ -module chain complex of  $\widetilde{M}$

$$\begin{aligned} \text{coker} \left( i_\alpha - zi_\beta : \mathbb{Z}[\pi_1(M)] \otimes_{\mathbb{Z}[\pi_1(N)]} C(\widetilde{N}) \rightarrow \mathbb{Z}[\pi_1(M)] \otimes_{\mathbb{Z}[\pi_1(M_1)]} C(\widetilde{M}_1) \right) \\ = C(\widetilde{M}) \end{aligned}$$

by the Mayer-Vietoris theorem.  $\square$

Let  $R *_{\alpha, \beta} \{z\}$  be an  $HNN$  extension of rings in which the morphisms  $\alpha, \beta : S \rightarrow R$  are both injections of  $(S, S)$ -bimodule direct summands, and  $R_\alpha, R_\beta$  are flat  $S$ -modules. (This is the case in the above example if  $\pi_1(N) \rightarrow \pi_1(M)$  is injective). Then the natural ring morphisms

$$\begin{aligned} R &\rightarrow R *_{\alpha, \beta} \{z\} , \quad S \rightarrow R *_{\alpha, \beta} \{z\} , \\ A &= \begin{pmatrix} R & R_\alpha \oplus R_\beta \\ 0 & S \end{pmatrix} \rightarrow \Sigma^{-1}A = M_2(R *_{\alpha, \beta} \{z\}) \end{aligned}$$

are injective, and  $\Sigma^{-1}A$  is a stably flat universal localization, with  $H(A, \Sigma) = \text{Nil}(R, S, \alpha, \beta)$  the nilpotent category of Waldhausen [30]. The chain complex lifting property of  $\Sigma^{-1}A$  gives a noncommutative localization proof of the existence of Mayer-Vietoris presentations for finite f.g. free  $R *_{\alpha, \beta} \{z\}$ -module chain complexes  $C$

$$0 \longrightarrow R *_{\alpha, \beta} \{z\} \otimes_S E \xrightarrow{i_{\alpha} - z i_{\beta}} R *_{\alpha, \beta} \{z\} \otimes_R D \longrightarrow C \longrightarrow 0$$

with  $D$  (resp.  $E$ ) a finite f.g. free  $R$ - (resp.  $S$ -) module chain complex ([30], Ranicki [23]). The algebraic  $K$ -theory localization exact sequence of [14]

$$\begin{aligned} \cdots \rightarrow K_{n+1}(A, \Sigma) &= K_n(S) \oplus K_n(S) \oplus \widetilde{\text{Nil}}_n(R, S, \alpha, \beta) \\ \begin{pmatrix} \alpha & \beta & 0 \\ 1 & 1 & 0 \end{pmatrix} &\xrightarrow{\quad} K_n(A) = K_n(R) \oplus K_n(S) \\ &\rightarrow K_n(\Sigma^{-1}A) = K_n(R *_{\alpha, \beta} \{z\}) \rightarrow \cdots \end{aligned}$$

is just the stabilization by  $1 : K_*(S) \rightarrow K_*(S)$  of the Mayer-Vietoris exact sequence of [30]

$$\cdots \longrightarrow K_n(S) \oplus \widetilde{\text{Nil}}_n(R, \alpha, \beta) \xrightarrow{(\alpha - \beta) \oplus 0} K_n(R) \longrightarrow K_n(R *_{\alpha, \beta} \{z\}) \longrightarrow \cdots$$

In particular, for  $\alpha = \beta = 1 : S = R \rightarrow R$  the  $HNN$  extension is just the Laurent polynomial extension

$$R *_{\alpha, \beta} \{z\} = R[z, z^{-1}]$$

and the Mayer-Vietoris exact sequence splits to give the original splitting of Bass, Heller and Swan [1]

$$K_1(R[z, z^{-1}]) = K_1(R) \oplus K_0(R) \oplus \widetilde{\text{Nil}}_0(R) \oplus \widetilde{\text{Nil}}_0(R)$$

as well as its extension to the Quillen higher  $K$ -groups  $K_*$ .

## 2.4 Amalgamated free products

The amalgamated free product  $R_1 *_S R_2$  is defined for any ring morphisms  $i_1 : S \rightarrow R_1$ ,  $i_2 : S \rightarrow R_2$ , with

$$r_1 i_1(s) * r_2 = r_1 * i_2(s) r_2 \in R_1 *_S R_2 \quad (r_1 \in R_1, r_2 \in R_2, s \in S).$$

Define the triangular  $3 \times 3$  matrix ring

$$A = \begin{pmatrix} R_1 & 0 & R_1 \\ 0 & R_2 & R_2 \\ 0 & 0 & S \end{pmatrix}$$

and the  $A$ -module morphisms

$$\begin{aligned} \sigma_1 &= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} : P_1 = \begin{pmatrix} R_1 \\ 0 \\ 0 \end{pmatrix} \rightarrow P_3 = \begin{pmatrix} R_1 \\ R_2 \\ S \end{pmatrix} , \\ \sigma_2 &= \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} : P_2 = \begin{pmatrix} 0 \\ R_2 \\ 0 \end{pmatrix} \rightarrow P_3 = \begin{pmatrix} R_1 \\ R_2 \\ S \end{pmatrix} . \end{aligned}$$

The  $A$ -modules  $P_1, P_2, P_3$  are f.g. projective since  $P_1 \oplus P_2 \oplus P_3 = A$ . The noncommutative localization of  $A$  inverting  $\Sigma = \{\sigma_1, \sigma_2\}$  is the full  $3 \times 3$  matrix ring

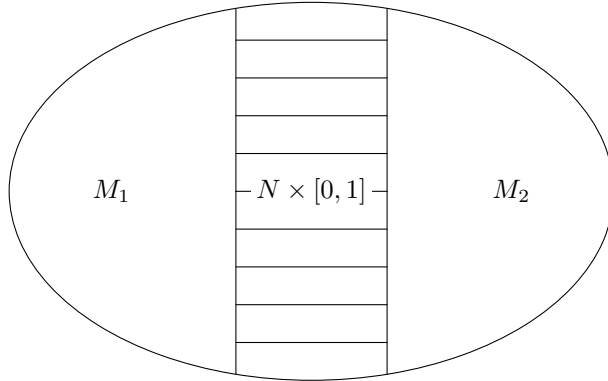
$$\Sigma^{-1}A = M_3(R_1 *_S R_2)$$

(a modification of Theorem 4.10 of [24]).

*Example* Let  $(M, N \subseteq M)$  be a separating pair of  $CW$  complexes such that  $N$  has a neighbourhood  $N \times [0, 1] \subseteq M$  and

$$M = M_1 \cup_{N \times \{0\}} N \times [0, 1] \cup_{N \times \{1\}} M_2$$

with  $M_1, M_2, N$  connected.



By the Seifert-van Kampen theorem, the fundamental group of  $M$  is the amalgamated free product

$$\pi_1(M) = \pi_1(M_1) *_{\pi_1(N)} \pi_1(M_2) ,$$

so that

$$\mathbb{Z}[\pi_1(M)] = \mathbb{Z}[\pi_1(M_1)] *_{\mathbb{Z}[\pi_1(N)]} \mathbb{Z}[\pi_1(M_2)] .$$

As above, define a triangular matrix ring

$$A = \begin{pmatrix} \mathbb{Z}[\pi_1(M_1)] & 0 & \mathbb{Z}[\pi_1(M_1)] \\ 0 & \mathbb{Z}[\pi_1(M_2)] & \mathbb{Z}[\pi_1(M_2)] \\ 0 & 0 & \mathbb{Z}[\pi_1(N)] \end{pmatrix}$$

with noncommutative localization

$$\Sigma^{-1}A = M_3(\mathbb{Z}[\pi_1(M_1)] *_{\mathbb{Z}[\pi_1(N)]} \mathbb{Z}[\pi_1(M_2)]) = M_3(\mathbb{Z}[\pi_1(M)]) .$$

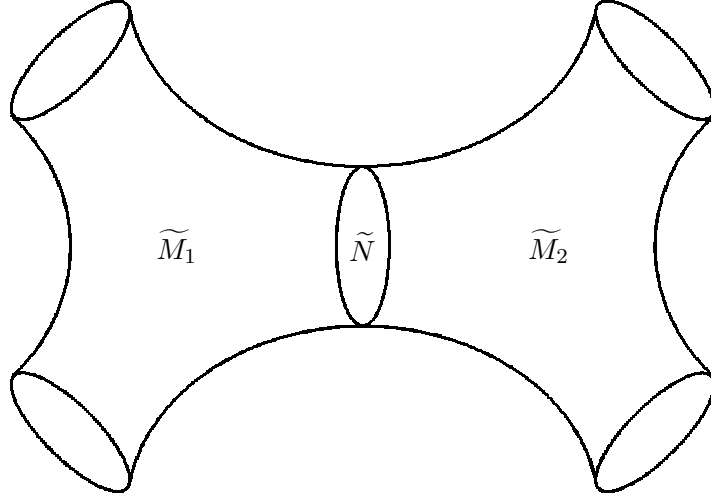
Assume that  $\pi_1(N) \rightarrow \pi_1(M)$  is injective, so that the morphisms

$$\begin{aligned} i_1 : \pi_1(N) &\rightarrow \pi_1(M_1) , \quad i_2 : \pi_1(N) \rightarrow \pi_1(M_2) , \\ \pi_1(M_1) &\rightarrow \pi_1(M) , \quad \pi_1(M_2) \rightarrow \pi_1(M) \end{aligned}$$

are all injective, and the universal cover  $\widetilde{M}$  of  $M$  is a union

$$\widetilde{M} = \bigcup_{g_1 \in [\pi_1(M) : \pi_1(M_1)]} g_1 \widetilde{M}_1 \cup \bigcup_{h \in [\pi_1(M) : \pi_1(N)]} h \widetilde{N} \cup \bigcup_{g_2 \in [\pi_1(M) : \pi_1(M_2)]} g_2 \widetilde{M}_2$$

of  $[\pi_1(M) : \pi_1(M_1)]$  translates of the universal cover  $\widetilde{M}_1$  of  $M_1$  and  $[\pi_1(M) : \pi_1(M_2)]$  translates of the universal cover  $\widetilde{M}_2$  of  $M_2$  with intersection the  $[\pi_1(M) : \pi_1(N)]$  translates of the universal cover  $\widetilde{N}$  of  $N$ .



The cellular f.g. free chain complexes  $C(\widetilde{M}_1)$ ,  $C(\widetilde{N})$  of the universal covers  $\widetilde{M}_1$ ,  $\widetilde{N}$  are related by  $\mathbb{Z}[\pi_1(M_1)]$ -module chain maps

$$\begin{aligned} i_1 &: \mathbb{Z}[\pi_1(M_1)] \otimes_{\mathbb{Z}[\pi_1(N)]} C(\widetilde{N}) \rightarrow C(\widetilde{M}_1) , \\ i_2 &: \mathbb{Z}[\pi_1(M_2)] \otimes_{\mathbb{Z}[\pi_1(N)]} C(\widetilde{N}) \rightarrow C(\widetilde{M}_2) \end{aligned}$$

defining a f.g. projective  $A$ -module chain complex  $\begin{pmatrix} C(\widetilde{M}_1) \\ C(\widetilde{M}_2) \\ C(\widetilde{N}) \end{pmatrix}$  with assembly the cellular f.g. free  $\mathbb{Z}[\pi_1(M)]$ -module chain complex of  $\widetilde{M}$

$$\begin{aligned} \text{coker} \left( \begin{pmatrix} 1 \otimes i_1 \\ 1 \otimes i_2 \end{pmatrix} : \mathbb{Z}[\pi_1(M)] \otimes_{\mathbb{Z}[\pi_1(N)]} C(\widetilde{N}) \rightarrow \right. \\ \left. \mathbb{Z}[\pi_1(M)] \otimes_{\mathbb{Z}[\pi_1(M_1)]} C(\widetilde{M}_1) \oplus \mathbb{Z}[\pi_1(M)] \otimes_{\mathbb{Z}[\pi_1(M_2)]} C(\widetilde{M}_2) \right) \\ = C(\widetilde{M}) \end{aligned}$$

by the Mayer-Vietoris theorem.  $\square$

Let  $R_1 *_S R_2$  be an amalgamated free product of rings in which the morphisms  $i_1 : S \rightarrow R_1$ ,  $i_2 : S \rightarrow R_2$  are both injections of  $(S, S)$ -bimodule direct summands, and  $R_1, R_2$  are flat  $S$ -modules. (This is the case in the above example if  $\pi_1(N) \rightarrow \pi_1(M)$  is injective). Then the natural ring morphisms

$$\begin{aligned} R_1 \rightarrow R_1 *_S R_2 , \quad R_2 \rightarrow R_1 *_S R_2 , \quad S \rightarrow R_1 *_S R_2 , \\ A = \begin{pmatrix} R_1 & 0 & R_1 \\ 0 & R_2 & R_2 \\ 0 & 0 & S \end{pmatrix} \rightarrow \Sigma^{-1}A = M_3(R_1 *_S R_2) \end{aligned}$$

are injective, and  $\Sigma^{-1}A$  is a stably flat noncommutative localization, with  $H(A, \Sigma) = \text{Nil}(R_1, R_2, S)$  the nilpotent category of Waldhausen [30]. The chain complex lifting property of  $\Sigma^{-1}A$  gives a noncommutative localization proof of the existence of Mayer-Vietoris presentations for finite f.g. free  $R_1 *_S R_2$ -module chain complexes  $C$

$$0 \longrightarrow R_1 *_S R_2 \otimes_S E \longrightarrow R_1 *_S R_2 \otimes_{R_1} D_1 \oplus R_1 *_S R_2 \otimes_{R_2} D_2 \longrightarrow C \longrightarrow 0$$

with  $D_i$  (resp.  $E$ ) a finite f.g. free  $R_i$ - (resp.  $S$ -) module chain complex ([30], Ranicki [23]). The algebraic  $K$ -theory localization exact sequence of



[14]

$$\begin{aligned}
\cdots \rightarrow K_{n+1}(A, \Sigma) &= K_n(S) \oplus K_n(S) \oplus \widetilde{\text{Nil}}_n(R_1, R_2, S) \\
\begin{pmatrix} i_1 & 0 & 0 \\ 0 & i_2 & 0 \\ 1 & 1 & 0 \end{pmatrix} &\longrightarrow K_n(A) = K_n(R_1) \oplus K_n(R_2) \oplus K_n(S) \\
&\rightarrow K_n(\Sigma^{-1}A) = K_n(R_1 *_S R_2) \rightarrow \cdots
\end{aligned}$$

is just the stabilization by  $1 : K_*(S) \rightarrow K_*(S)$  of the Mayer-Vietoris exact sequence of [30]

$$\begin{aligned}
\cdots \longrightarrow K_n(S) \oplus \widetilde{\text{Nil}}_n(R_1, R_2, S) \\
\begin{pmatrix} i_1 & 0 \\ i_2 & 0 \end{pmatrix} &\longrightarrow K_n(R_1) \oplus K_n(R_2) \longrightarrow K_n(R_1 *_S R_2) \longrightarrow \cdots
\end{aligned}$$

## 2.5 The algebraic $L$ -theory of a noncommutative localization

See Chapter 3 of Ranicki [19] for the algebraic  $L$ -theory of a commutative localization.

The algebraic  $L$ -theory of a ring  $A$  depends on an involution, that is a function  $\bar{\cdot} : A \rightarrow A; a \mapsto \bar{a}$  such that

$$\overline{a+b} = \bar{a} + \bar{b}, \quad \overline{ab} = \bar{b}\bar{a}, \quad \bar{\bar{a}} = a, \quad \bar{1} = 1 \quad (a, b \in A).$$

For an injective noncommutative localization  $A \rightarrow \Sigma^{-1}A$  of a ring  $A$  with an involution which extends to  $\Sigma^{-1}A$  Vogel [29] obtained a localization exact sequence in quadratic  $L$ -theory

$$\cdots \rightarrow L_n(A) \rightarrow L_n(\Sigma^{-1}A) \rightarrow L_n(A, \Sigma) \rightarrow L_{n-1}(A) \rightarrow \cdots$$

with  $L_n(A, \Sigma) = L_{n-1}(H(A, \Sigma))$ . (See [14] for the symmetric  $L$ -theory localization exact sequence in the stably flat case). At first sight, it does not appear possible to apply this sequence to the triangular matrix rings of sections 2.2, 2.3, 2.4. How does one define an involution on a triangular matrix ring

$$A = \begin{pmatrix} A_1 & B \\ 0 & A_2 \end{pmatrix} ?$$

The trick is to observe that if  $A_1, A_2$  are rings with involution, and  $(B, \beta)$  is a nonsingular symmetric form over  $A_1$  such that  $B$  is an  $(A_1, A_2)$ -bimodule then  $A$  has a *chain duality* in the sense of Definition 1.1 of Ranicki [20], sending an  $A$ -module  $M = \begin{pmatrix} M_1 \\ M_2 \end{pmatrix}$  to the 1-dimensional  $A$ -module chain complex

$$TM : TM_1 = \begin{pmatrix} M_1^* \\ 0 \end{pmatrix} \rightarrow TM_0 = \begin{pmatrix} B \otimes_{A_2} M_2^* \\ M_2^* \end{pmatrix} .$$

The quadratic  $L$ -groups of  $A$  are just the relative  $L$ -groups in the exact sequence

$$\cdots \rightarrow L_n(A) \rightarrow L_n(A_2) \xrightarrow{(B, \beta) \otimes_{A_2} -} L_n(A_1) \rightarrow L_{n-1}(A) \rightarrow \cdots .$$

In particular, for generalized free products of rings with involution the triangular matrix rings  $A$  of section 2.3, 2.4 have such chain dualities, and in the injective case the torsion  $L$ -groups  $L_*(A, \Sigma) = L_{*-1}(H(A, \Sigma))$  in the localization exact sequence

$$\cdots \rightarrow L_n(A) \rightarrow L_n(\Sigma^{-1}A) \rightarrow L_n(A, \Sigma) \rightarrow L_{n-1}(A) \rightarrow \cdots$$

are just the unitary nilpotent  $L$ -groups  $\text{UNil}_*$  of Cappell [2].

## References

- [1] H. Bass, A. Heller and R. Swan, *The Whitehead group of a polynomial extension*, Publ. Math. I.H.E.S. 22, 61–80 (1964)
- [2] S. Cappell, *Unitary nilpotent groups and hermitian  $K$ -theory*, Bull. A.M.S. 80, 1117–1122 (1974)
- [3] ——— and J. Shaneson, *The codimension two placement problem, and homology equivalent manifolds*, Ann. of Maths. 99, 277–348 (1974)
- [4] ——— and ———, *Link cobordism*, Comm. Math. Helv. 55, 20–49 (1980)
- [5] P. M. Cohn, *Free rings and their relations*, Academic Press (1971)
- [6] O. Cornea and A. Ranicki, *Rigidity and glueing for the Morse and Novikov complexes*, <http://arXiv.math.AT/0107221>, to appear in J. Eur. Math. Soc.

- [7] W. Dicks and E. Sontag, *Sylvester domains*, J. Pure Appl. Algebra 13, 243–275 (1978)
- [8] J. Duval, *Forme de Blanchfield et cobordisme d’entrelacs bords*, Comm. Math. Helv. 61, 617–635 (1986)
- [9] M. Farber, *Morse-Novikov critical point theory, Cohn localization and Dirichlet units*, Commun. Contemp. Math 1, 467–495 (1999)
- [10] ——— and A. Ranicki, *The Morse-Novikov theory of circle-valued functions and noncommutative localization*, <http://arXiv.math.DG/9812122>, Proc. 1998 Moscow Conference for S.P.Novikov’s 60th Birthday, Proc. Steklov Inst. 225, 381–388 (1999)
- [11] ——— and P. Vogel, *The Cohn localization of the free group ring*, Math. Proc. Camb. Phil. Soc. 111, 433–443 (1992)
- [12] S. Garoufalidis and A. Kriker, *A rational noncommutative invariant of boundary links*, <http://arXiv.math.GT/0105028> (2001)
- [13] B. Hughes and A. Ranicki, *Ends of complexes*, Cambridge Tracts in Mathematics 123, CUP (1996)
- [14] A. Neeman and A. Ranicki, *Noncommutative localization and chain complexes I. Algebraic K- and L-theory*, <http://arXiv.math.RA/0109118> (2001)
- [15] S. P. Novikov, *The hamiltonian formalism and a multi-valued analogue of Morse theory*, Russian Math. Surveys 37:5, 1–56 (1982)
- [16] A. Pajitnov, *Incidence coefficients in the Novikov complex for Morse forms: rationality and exponential growth properties*, <http://arXiv.math.dg-ga/9604004>, St. Petersburg Math. J. 9, 969 – 1006 (1998)
- [17] F. Quinn, *Open book decompositions, and the bordism of automorphisms*, Topology 18, 55–73 (1979)
- [18] ———, *Dual decompositions of 4-manifolds II. Linear link invariants*, <http://arXiv.math.GT/0109148> (2001)
- [19] A. Ranicki, *Exact sequences in the algebraic theory of surgery*, Mathematical Notes 26, Princeton (1981)

- [20] ——— , *Algebraic L-theory and topological manifolds*, Cambridge Tracts in Mathematics 102, CUP (1992)
- [21] ——— , *High dimensional knot theory*, Springer Mathematical Monograph, Springer (1998)
- [22] ——— , *The algebraic construction of the Novikov complex of a circle-valued Morse function*, <http://arXiv.math.AT/9903090>, Math. Ann. 322, 745–785 (2002)
- [23] ——— , *Algebraic and combinatorial codimension 1 transversality*, <http://arXiv.math.AT/0308111>
- [24] A. Schofield, *Representations of rings over skew fields*, LMS Lecture Notes 92, Cambridge (1985)
- [25] D. Sheiham, *Noncommutative characteristic polynomials and Cohn localization*, J. London Math. Soc. 64, 13–28 (2001)
- [26] ——— , *Invariants of boundary link cobordism*, Edinburgh Ph. D. thesis (2001) <http://arXiv.math.AT/0110249>, A.M.S. Memoir (to appear)
- [27] L. Siebenmann, *The obstruction to finding the boundary of an open manifold of dimension greater than five*, Princeton Ph.D. thesis (1965). <http://www.maths.ed.ac.uk/~aar/surgery/sieben.pdf>
- [28] P.Vogel, *On the obstruction group in homology surgery*, Publ. Math. I.H.E.S. 55, 165–206 (1982)
- [29] ——— , *Localisation non commutative de formes quadratiques*, Springer Lecture Notes 967, 376–389 (1982)
- [30] F. Waldhausen, *Algebraic K-theory of generalized free products*, Ann. of Maths. 108, 135–256 (1978)
- [31] C.T.C. Wall, *Finiteness conditions for CW complexes*, Ann. of Maths. 81, 55–69 (1965)
- [32] ——— , *Surgery on compact manifolds*, Academic Press (1970), 2nd edition A.M.S. (1999)

School of Mathematics  
University of Edinburgh  
James Clerk Maxwell Building

King's Buildings  
Mayfield Road  
Edinburgh EH9 3JZ  
SCOTLAND, UK

e-mail [aar@maths.ed.ac.uk](mailto:aar@maths.ed.ac.uk)